

一. 填空题

1. 极限 $\lim_{x \rightarrow 0} \frac{\sin 3x}{x} =$

解: $\lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \cdot 3 = 1 \cdot 3 = 3$

2. 极限 $\lim_{x \rightarrow 1} \frac{\cos \frac{\pi}{2}x}{x-1} =$

解: $\lim_{x \rightarrow 1} \frac{\cos \frac{\pi}{2}x}{x-1} = \lim_{x \rightarrow 1} \frac{\sin(\frac{\pi}{2} - \frac{\pi}{2}x)}{x-1} = \lim_{x \rightarrow 1} \frac{\sin \frac{\pi}{2}(1-x)}{\frac{\pi}{2}(1-x)} \cdot (-\frac{\pi}{2})$
 $= 1 \cdot (-\frac{\pi}{2}) = -\frac{\pi}{2}$

3. 极限 $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} =$

解: $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} = \lim_{x \rightarrow 0} 2 \cdot \frac{\sin x}{x} \cdot \frac{\sin x}{x}$
 $= 2 \cdot 1 \cdot 1 = 2$

4. 极限 $\lim_{x \rightarrow 0} (1-3x)^{\frac{1}{x}} =$

解: $\lim_{x \rightarrow 0} (1-3x)^{\frac{1}{x}} = \lim_{x \rightarrow 0} [1 + (-3x)]^{\frac{1}{-3x} \cdot (-3)}$
 $= e^{-3}$

二. 计算题 1. 求极限 $\lim_{n \rightarrow \infty} (\frac{1}{\sqrt{n^2+\pi}} + \frac{1}{\sqrt{n^2+2\pi}} + \dots + \frac{1}{\sqrt{n^2+n\pi}})$

解: $\because n \cdot \frac{1}{\sqrt{n^2+n\pi}} < \frac{1}{\sqrt{n^2+\pi}} + \frac{1}{\sqrt{n^2+2\pi}} + \dots + \frac{1}{\sqrt{n^2+n\pi}} < n \cdot \frac{1}{\sqrt{n^2+\pi}}$

$\lim_{n \rightarrow \infty} n \cdot \frac{1}{\sqrt{n^2+n\pi}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{\pi}{n}}} = 1$ $\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+\pi}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{\pi}{n^2}}} = 1$

$\therefore$ 由夹逼定理知, $\lim_{n \rightarrow \infty} (\frac{1}{\sqrt{n^2+\pi}} + \dots + \frac{1}{\sqrt{n^2+n\pi}}) = 1$

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2. 求极限 $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n} \right)$

解: $\frac{1}{n^2+n} + \frac{2}{n^2+n} + \cdots + \frac{n}{n^2+n} < \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n} < \frac{1}{n^2+1} + \frac{2}{n^2+1} + \cdots + \frac{n}{n^2+1}$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n^2+n} + \frac{2}{n^2+n} + \cdots + \frac{n}{n^2+n} \right) = \lim_{n \rightarrow \infty} \frac{\frac{1}{2} n(n+1)}{n^2+n} = \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n^2+1} + \frac{2}{n^2+1} + \cdots + \frac{n}{n^2+1} \right) = \lim_{n \rightarrow \infty} \frac{\frac{1}{2} n(n+1)}{n^2+1} = \lim_{n \rightarrow \infty} \frac{1}{2} \frac{1 + \frac{1}{n}}{1 + \frac{1}{n^2}} = \frac{1}{2}$$

由夹逼定理知, $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n} \right) = \frac{1}{2}$

3. 求极限 $\lim_{n \rightarrow \infty} (1+3^n+5^n)^{\frac{1}{n}}$

解: $(5^n)^{\frac{1}{n}} < (1+3^n+5^n)^{\frac{1}{n}} < (5^n+5^n+5^n)^{\frac{1}{n}} = 3^{\frac{1}{n}} \cdot 5$

$$\lim_{n \rightarrow \infty} (5^n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} 5 = 5$$

$$\lim_{n \rightarrow \infty} 3^{\frac{1}{n}} \cdot 5 = \lim_{n \rightarrow \infty} 3^{\frac{1}{n}} \cdot \lim_{n \rightarrow \infty} 5 = 5$$

由夹逼定理知, $\lim_{n \rightarrow \infty} (1+3^n+5^n)^{\frac{1}{n}} = 5$

4. 求极限 $\lim_{x \rightarrow 1} \frac{\sin \pi x}{x-1}$

解: $\lim_{x \rightarrow 1} \frac{\sin \pi x}{x-1} = \lim_{x \rightarrow 1} \frac{\sin(\pi - \pi x)}{x-1} = \lim_{x \rightarrow 1} \frac{\sin \pi(1-x)}{x-1}$

$$= \lim_{x \rightarrow 1} \frac{\sin \pi(1-x)}{\pi(1-x)} \cdot (-\pi) = 1 \cdot (-\pi)$$

$$= -\pi$$

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5. 求极限 $\lim_{x \rightarrow 0} \frac{x - \sin 2x}{x + \sin 2x}$

$$\begin{aligned} \text{解: } \lim_{x \rightarrow 0} \frac{x - \sin 2x}{x + \sin 2x} &= \lim_{x \rightarrow 0} \frac{1 - \frac{\sin 2x}{x}}{1 + \frac{\sin 2x}{x}} = \frac{1 - \lim_{x \rightarrow 0} \frac{\sin 2x}{x}}{1 + \lim_{x \rightarrow 0} \frac{\sin 2x}{x}} \\ &= \frac{1 - 2}{1 + 2} = -\frac{1}{3} \end{aligned}$$

6. 求极限 $\lim_{n \rightarrow \infty} \left(\cos \frac{x}{2} \cos \frac{x}{4} \cdots \cos \frac{x}{2^{n-1}} \cos \frac{x}{2^n} \right) \quad (x \neq 0)$

$$\begin{aligned} \text{解: } \lim_{n \rightarrow \infty} \left(\cos \frac{x}{2} \cos \frac{x}{4} \cdots \cos \frac{x}{2^{n-1}} \cos \frac{x}{2^n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{\cos \frac{x}{2} \cos \frac{x}{4} \cdots \cos \frac{x}{2^{n-1}} \cos \frac{x}{2^n} \cdot \sin \frac{x}{2^n}}{\sin \frac{x}{2^n}} \\ &= \lim_{n \rightarrow \infty} \frac{\cos \frac{x}{2} \cos \frac{x}{4} \cdots \cos \frac{x}{2^{n-1}} \cdot \frac{1}{2} \sin \frac{x}{2^{n-1}}}{\sin \frac{x}{2^n}} \\ &= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{2}\right)^n \sin x}{\sin \frac{x}{2^n}} = \lim_{n \rightarrow \infty} \frac{\frac{x}{2^n}}{\sin \frac{x}{2^n}} \cdot \frac{\sin x}{x} = 1 \cdot \frac{\sin x}{x} = \frac{\sin x}{x} \end{aligned}$$

7. 求极限 $\lim_{n \rightarrow \infty} n [\ln(n+2) - \ln n]$

$$\begin{aligned} \text{解: } \lim_{n \rightarrow \infty} n [\ln(n+2) - \ln n] \\ &= \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{2}{n} \right) \\ &= \lim_{n \rightarrow \infty} \ln \left(1 + \frac{2}{n} \right)^n \\ &= \lim_{n \rightarrow \infty} \ln \left(1 + \frac{2}{n} \right)^{\frac{n}{2} \cdot 2} \\ &= \ln e^2 \\ &= 2 \end{aligned}$$

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8. 求极限 $\lim_{x \rightarrow \infty} \left(\frac{x+2}{x+1} \right)^{3x+2}$

$$\begin{aligned} \text{解: } \lim_{x \rightarrow \infty} \left(\frac{x+2}{x+1} \right)^{3x+2} &= \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x+1} \right)^{(x+1) \cdot \frac{3x+2}{x+1}} \\ &= e^{\lim_{x \rightarrow \infty} \frac{3x+2}{x+1}} = e^3 \end{aligned}$$

9. 设极限 $\lim_{x \rightarrow \infty} \left(\frac{x+2a}{x-a} \right)^x = 8$, 求 a

$$\begin{aligned} \text{解: } \lim_{x \rightarrow \infty} \left(\frac{x+2a}{x-a} \right)^x &= \lim_{x \rightarrow \infty} \left(1 + \frac{3a}{x-a} \right)^{\frac{x-a}{3a} \cdot \frac{3ax}{x-a}} \\ &= e^{\lim_{x \rightarrow \infty} \frac{3ax}{x-a}} = e^{3a} = 8 \end{aligned}$$

$$\Rightarrow 3a = \ln 8 = 3 \ln 2 \Rightarrow a = \ln 2$$

注. $\lim_{x \rightarrow ?} f(x)^{g(x)} = \lim_{x \rightarrow ?} f(x)^{\lim_{x \rightarrow ?} g(x)}$ (如果 $\lim_{x \rightarrow ?} f(x) \neq 0$ 和 $\neq 1$)

10. $\lim_{x \rightarrow 0} (1-3x)^{\frac{2}{x}}$

$$\text{解: } \lim_{x \rightarrow 0} (1-3x)^{\frac{2}{x}} = \lim_{x \rightarrow 0} [1 + (-3x)]^{\frac{1}{-3x} \cdot \frac{(-3x) \cdot 2}{x}} = e^{\lim_{x \rightarrow 0} \frac{-6x}{x}} = e^{-6}$$

11. 求极限 $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$

解: 设 $e^x - 1 = h(x)$ 则 $e^x = 1 + h(x) \Rightarrow x = \ln[1 + h(x)]$

$$\begin{aligned} \text{于是, } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} &= \lim_{x \rightarrow 0} \frac{1}{\frac{x}{e^x - 1}} \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{1}{h(x)} \ln[1 + h(x)]} = \lim_{x \rightarrow 0} \frac{1}{\ln[1 + h(x)] \cdot \frac{1}{h(x)}} \\ &= \frac{1}{\ln e} = 1 \end{aligned}$$

注, $x \rightarrow 0$ 时 $h(x) \rightarrow 0$

三. 证明题

$$1. \lim_{x \rightarrow 0^+} x \left[\frac{1}{x} \right] = 1$$

证明: $x \rightarrow 0^+$ 则 $\frac{1}{x} \rightarrow +\infty$, 显然有 $\left[\frac{1}{x} \right] \leq \frac{1}{x}$ 且 $\left[\frac{1}{x} \right] > \frac{1}{x} - 1$

即

$$x \cdot \left(\frac{1}{x} - 1 \right) < x \left[\frac{1}{x} \right] \leq x \cdot \frac{1}{x} = 1$$

$$\text{又 } \lim_{x \rightarrow 0^+} x \cdot \left(\frac{1}{x} - 1 \right) = \lim_{x \rightarrow 0^+} (1 - x) = 1 - 0 = 1$$

由夹逼定理知 $\lim_{x \rightarrow 0^+} x \left[\frac{1}{x} \right] = 1$

2. 设 $x_1 = 10$, $x_{n+1} = \sqrt{6 + x_n}$ ($n \geq 1$), 证明数列 $\{x_n\}$ 收敛, 并求其极限

证明: 先用数学归纳法证明数列 $\{x_n\}$ 是单调递减的.

$$\textcircled{1} \text{显然 } x_1 = 10 > x_2 = \sqrt{6 + x_1} = \sqrt{6 + 10} = 4$$

$$\textcircled{2} \text{不妨假设 } x_k < x_{k-1} < x_{k-2} < \dots < x_2 < x_1$$

则 $x_k + 6 < x_{k-1} + 6$ 即 $\sqrt{x_k + 6} < \sqrt{x_{k-1} + 6}$, 于是 $x_{k+1} < x_k$

由 $\textcircled{1}\textcircled{2}$ 知数列 $\{x_n\}$ 是单调递减的。显然 $x_{n+1} = \sqrt{6 + x_n} > 0$

由单调有界数列必有极限知数列 $\{x_n\}$ 收敛, 设 $\lim_{n \rightarrow \infty} x_n = A$

$$\text{则 } \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \sqrt{6 + x_n} \Rightarrow A = \sqrt{6 + A} \Rightarrow A = 3$$

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3. 设 $y_n \leq x_n \leq z_n$, 且 $\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = a$, 证明: $\lim_{n \rightarrow \infty} x_n = a$

证明: $\lim_{n \rightarrow \infty} y_n = a \Rightarrow \forall \varepsilon > 0, \exists N_1, \text{当 } n > N_1 \text{ 时 } |y_n - a| < \varepsilon$

$\lim_{n \rightarrow \infty} z_n = a \Rightarrow \text{对于上面取定的 } \varepsilon, \exists N_2, \text{当 } n > N_2 \text{ 时 } |z_n - a| < \varepsilon$

取 $N = \max\{N_1, N_2\}$, 则当 $n > N$ 时, $|y_n - a| < \varepsilon$ 与 $|z_n - a| < \varepsilon$

同时成立, 即 $a - \varepsilon < y_n < a + \varepsilon$

$$a - \varepsilon < z_n < a + \varepsilon$$

由于 $y_n \leq x_n \leq z_n$, 则 $a - \varepsilon < x_n < a + \varepsilon$

$$\therefore \lim_{n \rightarrow \infty} x_n = a$$

用定义证明数列的极限一定要理解极限的定义